MTH 166

Lecture-28

Divergence and Curl of a Vector Field

Topic:

Vector Differential Calculus

Learning Outcomes:

1. To calculate divergence of a vector field

2. To calculate curl of a vector field

Let the given vector field be $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

Let us consider a vector differential operator $\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

<u>Divergence:</u> It is dot product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial x} \qquad (\because \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1)$$

Divergence is always a scalar quantity (Having no direction).

<u>Note:</u> If $\mathbf{div}(\vec{\boldsymbol{v}}) = \mathbf{0}$, then vector $\vec{v}$ is called **Solenoidal** or **Incompressible**.

That means the field has no sources and no sinks.

Curl: It is cross product between $\vec{\nabla}$ and $\vec{v}$ and is given by:

$$\operatorname{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Curl is always a vector quantity.

Note: If $\mathbf{curl}(\vec{\boldsymbol{v}}) = \vec{\mathbf{0}}$, then vector $\vec{v}$ is called **Irrotational** or **Conservative**.

Two Important Properties:

1. div(curl($\vec{v}$)) = 0 (Always)

2. curl(grad (f)) = $\vec{0}$ (Always)

Problem 1. Compute div($\vec{v}$) and curl($\vec{v}$) and verify that div(curl($\vec{v}$)) = 0 where

$\vec{v} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

Solution. Here $\vec{v} = x\hat{\imath} + 2y\hat{\jmath} + z\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$$v_1 = x, \quad v_1 = 2y, \quad v_1 = z$$

$$\Rightarrow \frac{\partial v_1}{\partial x} = 1, \quad \frac{\partial v_2}{\partial y} = 2, \quad \frac{\partial v_3}{\partial z} = 1$$

$$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial x}$$

$$= 1 + 2 + 1 = 4$$

$$\text{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(z)}{\partial y} - \frac{\partial(2y)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z}\right) + \hat{k}\left(\frac{\partial(2y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)$$

$$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0} \ \ \text{(Irrotational)}$$

$$\text{div}(\text{curl}(\vec{v})) = \text{div}(\vec{0}) = 0$$

Problem 2. Compute div($\vec{v}$) and curl($\vec{v}$) and verify that div(curl($\vec{v}$)) = 0 where

$\vec{v} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$

Solution. Here $\vec{v} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$

Comparing with: $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$

$$v_1 = xy, \quad v_1 = yz, \quad v_1 = zx$$

$$\Rightarrow \frac{\partial v_1}{\partial x} = y, \quad \frac{\partial v_2}{\partial y} = z, \quad \frac{\partial v_3}{\partial z} = x$$

$$\text{div}(\vec{v}) = \vec{\nabla}.\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial x}$$

$$= y + z + x$$

$$\mathrm{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(zx)}{\partial y} - \frac{\partial(yz)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(zx)}{\partial x} - \frac{\partial(xy)}{\partial z}\right) + \hat{k}\left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)$$

$$= -y\hat{\imath} - z\hat{\jmath} - x\hat{k}$$

$$\mathrm{div}(\mathrm{curl}(\vec{v})) = \mathrm{div}(-y\hat{\imath} - z\hat{\jmath} - x\hat{k}) = \frac{\partial(-y)}{\partial x} + \frac{\partial(-z)}{\partial y} + \frac{\partial(-x)}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

Problem 3. Compute grad (f) and verify that curl(grad (f)) = $\vec{0}$ where $f = x + y - 2z^2$

Solution. Let $f(x, y, z) = x + y - 2z^2$

$\Rightarrow f_x = 1, f_y = 1, f_z = -4z$

Then gradient of scalar field f is given by:

$$\text{grad}(f) = \vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}\right) = \hat{\imath} + \hat{\jmath} - 4z\hat{k}$$

$$\text{curl}(\text{grad}\,(f)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 1 & -4z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

Problem 4. Compute grad (f) and verify that curl(grad (f)) = $\vec{0}$ where $f = 16xy^3z^2$

Solution. Let $f(x, y, z) = 16xy^3z^2$

$\Rightarrow f_x = 16y^3z^2, f_y = 48xy^2z^2, f_z = 32xy^3z$

Then gradient of scalar field f is given by:

$$\text{grad}(f) = \vec{\nabla} f = \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right) = 16y^3z^2\hat{\imath} + 48xy^2z^2\hat{\jmath} + 32xy^3z\hat{k}$$

$$\text{curl}(\text{grad}\,(f)) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 16y^3z^2 & 48xy^2z^2 & 32xy^3z \end{vmatrix} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$$